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On dense subsets of boundaries of Coxeter groups

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The purpose of this note is to introduce some results of recent papers [7], [8] and [9] about dense subsets of boundaries of Coxeter groups.

A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(i) $m(s,t) = m(t,s)$ for any $s, t \in S$,

(ii) $m(s,s) = 1$ for any $s \in S$, and

(iii) $m(s,t) \geq 2$ for any $s, t \in S$ such that $s \neq t$.

The pair (W, S) is called a *Coxeter system*. Let (W, S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. A subset $T \subset S$ is called a *spherical subset of* S, if the parabolic subgroup W_T is finite. For each $w \in W$, we define $S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$, where $\ell(w)$ is the minimum length of word in S which represents w. For a subset $T \subset S$, we also define $W^T = \{w \in W \mid S(w) = T\}$.

Let (W, S) be a Coxeter system and let S^f be the family of spherical subsets of S. We denote WS^f as the set of all cosets of the form wW_T, with $w \in W$ and $T \in S^f$. The sets S^f and WS^f are partially ordered by inclusion. Contractible simplicial complexes $K(W, S)$ and $\Sigma(W, S)$ are defined as the geometric realizations of the partially ordered sets S^f and WS^f, respectively ([4]). The natural embedding $S^f \to WS^f$ defined by $T \mapsto W_T$ induces an embedding $K(W, S) \to \Sigma(W, S)$ which we regard as an inclusion. The group W acts on $\Sigma(W, S)$ via simplicial automorphism. Then $\Sigma(W, S) = WK(W, S)$

and $\Sigma(W,S)/W \cong K(W,S)$ ([4]). For each $w \in W$, $wK(W,S)$ is called a *chamber* of $\Sigma(W,S)$. If W is infinite, then $\Sigma(W,S)$ is noncompact. In [10], G. Moussong proved that a natural metric on $\Sigma(W,S)$ satisfies the CAT(0) condition. Hence, if W is infinite, $\Sigma(W,S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W,S)$ ([4], [3]). This boundary $\partial\Sigma(W,S)$ is called the *boundary of* (W,S). We note that the natural action of W on $\Sigma(W,S)$ is properly discontinuous and cocompact ([4]), and this action induces an action of W on $\partial\Sigma(W,S)$.

A subset A of a space X is said to be *dense* in X, if $\overline{A} = X$. A subset A of a metric space X is said to be *quasi-dense*, if there exists $N > 0$ such that each point of X is N-close to some point of A.

Let (W,S) be a Coxeter system. Then W has the *word metric* d_ℓ defined by $d_\ell(w,w') = \ell(w^{-1}w')$ for each $w,w' \in W$.

In [7], the following theorems were proved.

Theorem 1. *Let (W,S) be a Coxeter system. Suppose that $W^{\{s_0\}}$ is quasi-dense in W with respect to the word metric and $o(s_0t_0) = \infty$ for some $s_0, t_0 \in S$, where $o(s_0t_0)$ is the order of s_0t_0 in W. Then there exists $\alpha \in \partial\Sigma(W,S)$ such that the orbit $W\alpha$ is dense in $\partial\Sigma(W,S)$.*

Suppose that a group Γ acts properly and cocompactly by isometries on a CAT(0) space X. Every element $\gamma \in \Gamma$ such that $o(\gamma) = \infty$ is a hyperbolic transformation of X, i.e., there exists a geodesic axis $c : \mathbb{R} \to X$ and a real number $a > 0$ such that $\gamma \cdot c(t) = c(t+a)$ for each $t \in \mathbb{R}$ ([3]). Then, for all $x \in X$, the sequence $\{\gamma^i x\}$ converges to $c(\infty)$ in $X \cup \partial X$. We denote $\gamma^\infty = c(\infty)$.

Theorem 2. *Let (W,S) be a Coxeter system. If the set*

$$\bigcup\{W^{\{s\}} \mid s \in S \text{ such that } o(st) = \infty \text{ for some } t \in S\}$$

is quasi-dense in W, then $\{w^\infty \mid w \in W \text{ such that } o(w) = \infty\}$ is dense in $\partial\Sigma(W,S)$.

Remark. For a negatively curved group G and the boundary ∂G of G,

(1) we can show that $G\alpha$ is dense in ∂G for each $\alpha \in \partial G$ by an easy argument, and

(2) it is known that $\{g^\infty \mid g \in G \text{ such that } o(g) = \infty\}$ is dense in ∂G ([2]).

As an application of Theorems 1 and 2, we obtained the following theorem in [7].

Theorem 3. *Let (W, S) be a Coxeter system. Suppose that there exist a maximal spherical subset T of S and an element $s_0 \in S$ such that $o(s_0 t) \geq 3$ for each $t \in T$ and $o(s_0 t_0) = \infty$ for some $t_0 \in T$. Then*

(1) *$W\alpha$ is dense in $\partial\Sigma(W, S)$ for some $\alpha \in \partial\Sigma(W, S)$, and*

(2) *$\{w^\infty \mid w \in W$ such that $o(w) = \infty\}$ is dense in $\partial\Sigma(W, S)$.*

Example. The Coxeter system defined by the diagram in Figure 1 is not hyperbolic in Gromov sense, since it contains a copy of $\mathbb{Z}^2$, and it satisfies the condition of Theorem 3.

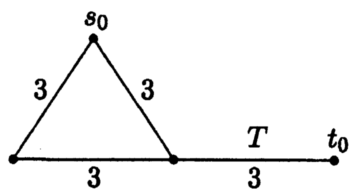

FIGURE 1

Suppose that a group G acts on a compact metric space X by homeomorphisms. Then X is said to be *minimal*, if every orbit Gx is dense in X.

For a negatively curved group G and the boundary ∂G of G, $G\alpha$ is dense in ∂G for each $\alpha \in \partial G$, that is, ∂G is minimal.

We note that Coxeter groups are non-positive curved groups and not negatively curved groups in general. There exist examples of Coxeter systems whose boundaris are not minimal as follows.

Example. Let $S = \{s, t, u\}$ and let

$$W = \langle\, S \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (us)^3 = 1 \,\rangle.$$

Then (W, S) is a Coxeter system and $\Sigma(W, S)$ is the flat Euclidean plane. For any $\alpha \in \partial\Sigma(W, S)$, $W\alpha$ is a finite-points set and not dense in $\partial\Sigma(W, S)$ which is a circle. This example implies that we can not omit the assumption "$m(s_0, t_0) = \infty$" in Theorem 3.

Example. Let $S = \{s_1, s_2, s_3, s_4\}$ and let

$$W = \langle\, S \mid s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1s_2)^2 = (s_2s_3)^2 = (s_3s_4)^2 = (s_4s_1)^2 = 1 \,\rangle.$$

Then (W, S) is a Coxeter system and $\Sigma(W, S)$ is the Euclidean plane. For any $\alpha \in \partial\Sigma(W, S)$, $W\alpha$ is a finite-points set and not dense in $\partial\Sigma(W, S)$ which is a circle. Here we note that $\{s_1, s_2\}$ is a maximal spherical subset of S, $m(s_1, s_3) = \infty$ and $m(s_2, s_3) = 2$. This example implies that we can not omit the assumption "$m(s_0, t) \geq 3$" in Theorem 3.

As an extension of Theorem 3, we have obtained the following theorem in [9].

Theorem 4. *Let (W, S) be a Coxeter system which satisfies the condition in Theorem 3. Then every orbit $W\alpha$ is dense in $\partial\Sigma(W, S)$, that is, $\partial\Sigma(W, S)$ is minimal.*

Here the following problems are open.

Problem. Does there exist a Coxeter system (W, S) such that some orbit $W\alpha$ is dense in $\partial\Sigma(W, S)$ and $\partial\Sigma(W, S)$ is not minimal?

Problem. Suppose that a group G acts geometrically on two CAT(0) spaces X and X'. Is it the case that ∂X is minimal if and only if $\partial X'$ is minimal?

Problem (Ruane). Suppose that a group G acts geometrically on a CAT(0) space X. Is it always the case that the set $\{g^\infty \mid g \in G, o(g) = \infty\}$ is dense in ∂X?

In [8], we also have obtained the following theorem.

Theorem 5. *Let (W, S) be a Coxeter system and let T be a subset of S such that W_T is infinite. If the set*

$$\bigcup\{W^{\{s\}} \mid s \in S \text{ such that } o(ss_0) = \infty \text{ and } s_0t \neq ts_0 \\ \text{for some } s_0 \in S \setminus T \text{ and } t \in \tilde{T}\}$$

is quasi-dense in W with respect to the word metric, then $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$, where $W_{\tilde{T}}$ is the essential parabolic subgroup of (W_T, T).

If W is a hyperbolic Coxeter group, then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$ for any $T \subset S$ such that W_T is infinite.

As an application of Theorem 5, we have obtained the following corollary in [8].

Corollary 6. *Let (W,S) be a Coxeter system and let T be a subset of S such that W_T is infinite. Suppose that there exist a maximal spherical subset U of S and an element $s \in S$ such that $o(su) \geq 3$ for every $u \in U$ and $o(su_0) = \infty$ for some $u_0 \in U$. If*

(1) *$s \notin T$ and $u_0 \in \tilde{T}$, or*
(2) *$u_0 \notin T$ and $s \in \tilde{T}$,*

then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$.

Concerning W-invariantness of $\partial\Sigma(W_T,T)$, the following theorem is known.
Theorem 7 ([6]).

(1) *Let (W,S) be a Coxeter system and $T \subset S$. Then $\partial\Sigma(W_T,T)$ is W-invariant if and only if $W = W_{\tilde{T}} \times W_{S \setminus \tilde{T}}$.*
(2) *Let (W,S) be an irreducible Coxeter system and let T be a proper subset of S such that W_T is infinite. Then $\partial\Sigma(W_T,T)$ is not W-invariant.*

Here the following problem is open.

Problem. Let (W,S) be a Coxeter system and let T be a subset of S such that W_T is infinite. Is it the case that if $\partial\Sigma(W_T,T)$ is not W-invariant then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$? Particulary, is it the case that if (W,S) is an irreducible Coxeter system then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$ for any subset T of S such that W_T is infinite?

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